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Vectorial exceptional families of elements

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Abstract Let $\mathcal{Z}$ be an ordered Hausdorff topological vector space with a preorder defined by a pointed closed convex cone $C \subset \mathcal{Z}$ with a nonempty interior. In this paper, we introduce exceptional families of elements w.r.t. C for multivalued mappings defined on a closed convex cone of a normed space X with values in the set $L(X, \mathcal{Z})$ of all continuous linear mappings from X into $\mathcal{Z}$. In Banach spaces, we prove a vectorial analogue of a theorem due to Bianchi, Hadjisavvas and Schaible. As an application, the C-EFE acceptability of C-pseudomonotone multivalued mappings is investigated.

Keywords Vector complementarity problems · Exceptional families of elements · Hemicontinuity · Pseudomonotone multivalued mappings

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1 Introduction

Throughout this paper, all topological vector spaces are assumed to be real and Hausdorff. For given topological vector spaces X and Y, we use $L(X, Y)$ for the set of all continuous linear mappings from X into Y. As usual, the set of all continuous linear functionals on X is written by X^*. For $\ell \in L(X, Y)$, let $\langle \ell, x \rangle$ denote the value of ℓ at $x \in X$.

Given any nonempty set S, let 2^S denote the set of all subsets of S. A mapping T from a nonempty set A into 2^S will be called a multivalued mapping from A into S. The graph of T is defined by $\mathcal{G}_T = \{(x, y) : x \in A \text{ and } y \in T(x)\}$.

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To formulate vector variational inequalities, we fix once and for all an ordered topological vector space $\mathcal{Z}$ with a preorder defined by a nonempty pointed closed convex cone $C \subset \mathcal{Z}$ with interior $\operatorname{int} C \neq \emptyset$.

Let Ω be a nonempty convex subset of a topological vector space X, and let T be a multivalued mapping from Ω into $L(X, \mathcal{Z})$ with nonempty values. The generalized vector variational inequality associated with (T, Ω, C), denoted by GVVI(T, Ω), is the problem to find a pair $(x, y) \in \mathcal{G}_T$ such that

$$\langle y, u - x\rangle \in (-\operatorname{int} C)^c \text{ for all } u \in \Omega,$$

where $(-\operatorname{int} C)^c = \mathcal{Z} \setminus (-\operatorname{int} C)$. Such a pair (x, y) is called a solution of the problem GVVI(T, Ω). When T is single-valued, this problem becomes the vector variational inequality VVI(T, Ω) which is to find $x \in \Omega$ such that $\langle T(x), u - x\rangle \in (-\operatorname{int} C)^c$ for all $u \in \Omega$. In this case, x is called a solution of VVI(T, Ω).

When $\Omega = \mathbb{K}$ is a closed convex cone in X, the problem GVVI(T, Ω) becomes the multivalued vector complementarity problem MVCP$(T, \mathbb{K})$. On the other hand, if T is single-valued, the corresponding problem is called the vector complementarity problem, denoted by VCP$(T, \mathbb{K})$.

When $\mathcal{Z} = \mathbb{R}$ and $C = \mathbb{R}_+ = \{r \in \mathbb{R} : r \geq 0\}$, the problem GVVI$(T, \Omega)$ becomes the generalized variational inequality GVI(T, Ω), i.e., the problem to find a pair $(x, y) \in \mathcal{G}_T$ such that $\langle y, u - x\rangle \geq 0$ for all $u \in \Omega$. Also, in this case, the problem MVCP$(T, \mathbb{K})$ becomes the multivalued complementarity problem MCP$(T, \mathbb{K})$.

The vector complementarity problem was first introduced and studied by Chen and Yang [5]. In the literature, most works deal with solvability and feasibility of vector complementarity problems; see [5,13,14,22,25] and references therein. The main purpose of this paper is to introduce the notion of exceptional family of elements to multivalued vector complementarity problems. See Sect. 3 for the definition of exceptional families of elements.

The notion of exceptional family of elements was originated from the notion of exceptional sequence of elements introduced by Smith [24], who proved that the complementarity problem associated with a continuous function $T : \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n$ is solvable when there is no exceptional sequences of elements for $(T, \mathbb{R}^n_+)$ [24, Corollary 4.4]. In 1997, by using topological degree, Isac et al. [21] introduced the notion of exceptional family of elements for continuous functions $T : \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n$, and proved that either the corresponding complementarity problem is solvable or there is an exceptional family of elements for $(T, \mathbb{R}^n_+)$. Since then this notion was further extended to complementarity problems in infinite dimensional spaces, and the existence of exceptional families of elements was intensively studied in the literature by several authors; see e.g., [4,6,7,17,19–21]. For a full discussion, see [15,16] and references therein.

Let X be a normed space. Given any nonempty set $\Omega \subset X$ and $r > 0$, we write

$$\Omega_r = \{x \in \Omega : \|x\| \leq r\}.$$

It follows from [4, Theorem 3.1] that if T is a multivalued mapping from a closed convex cone $\mathbb{K} \subset X$ into X^* with nonempty values, and if for every $r > 0$ the problem GVI$(T, \mathbb{K}_r)$ is solvable, then either the problem MCP$(T, \mathbb{K})$ is solvable or there is an exceptional family of elements for $(T, \mathbb{K})$. Our main work is to prove an analogue of [4, Theorem 3.1]; see Theorem 3.1 in Sect. 3. *From now on, unless stated otherwise, we shall use X for a normed space.*

The rest of this paper is organized as follows. In Sect. 2, we setup some preliminaries that we need in the sequel. Section 3 is used to define exceptional families of elements w.r.t. C for a multivalued mapping defined on a closed convex cone in X with values in $L(X, \mathcal{Z})$. As an application of our main result (Theorem 3.1), in Sect. 4, we prove that under some continuity assumptions, a C-pseudomonotone multivalued mapping T from a closed convex cone $\mathbb{K}$ in reflexive Banach space X into $L(X, \mathcal{Z})$ is C-EFE acceptable (Theorem 4.5), i.e., either the problem MVCP(T, $\mathbb{K}$) is solvable, or there exists a C-exceptional family of elements for $(T, \mathbb{K})$.

2 Vectorial dual cones

Given a nonempty convex set $\Omega \subset X$, we associate to each point $x \in \Omega$ a cone in $L(X, \mathcal{Z})$ given by

$$\Omega^*(C, x) = \{y \in L(X, \mathcal{Z}) : \langle y, u - x\rangle \in (-\operatorname{int} C)^c \text{ for all } u \in \Omega\},$$

called the C-dual cone of Ω at x. Observe that $\Omega^*(C, x)$ is closed in the simple convergence topology on $L(X, \mathcal{Z})$. See Sect. 4 for the definition of simple convergence topology. It follows immediately from the definition that if T is a multivalued mapping from Ω into $L(X, \mathcal{Z})$ with nonempty values, then a pair $(x, y) \in \mathcal{G}_T$ is a solution of the problem GVVI(T, Ω) if and only if $y \in T(x) \cap \Omega^*(C, x)$.

If $\mathbb{K}$ is a convex cone in X, then $\mathbb{K}^*(C, 0)$ coincides with the weak C-dual cone of $\mathbb{K}$ given in [25]. Observe that for every $x \in \mathbb{K}$,

(i) $\{y \in L(X, \mathcal{Z}) : \langle y, x\rangle \in (-C)\} \cap \mathbb{K}^*(C, 0) \subset \mathbb{K}^*(C, x)$;
(ii) $\mathbb{K}^*(C, x) \subset \{y \in L(X, \mathcal{Z}) : \langle y, x\rangle \in (\operatorname{int} C)^c \cap (-\operatorname{int} C)^c\} \cap \mathbb{K}^*(C, 0)$.

When $\mathcal{Z} = \mathbb{R}$, the $\mathbb{R}_+$-dual cones will be simply written by $\Omega^*(x)$. More precisely,

$$\Omega^*(x) = \{y \in X^* : \langle y, u - x\rangle \geq 0 \text{ for all } u \in \Omega\}.$$

Note that if $x \in \mathbb{K}$, then

$$\mathbb{K}^*(x) = \{y \in X^* : \langle y, x\rangle = 0 \text{ and } \langle y, u\rangle \geq 0 \text{ for all } u \in \mathbb{K}\}.$$

Theorem 2.1 *Let Ω be a nonempty convex subset of X, let $x_0 \in \Omega$, and let $y \in \Omega_r^*(C, x_0)$ with $r > 0$. If $\langle y, u - x_0\rangle = 0$ for some $u \in \Omega$ with $\|u\| < r$, then $y \in \Omega^*(C, x_0)$.*

Proof It suffices to prove that $\langle y, x - x_0\rangle \in (-\operatorname{int} C)^c$ for all $x \in \Omega \setminus \Omega_r$. For $0 \leq t \leq 1$, write $u_t = (1-t)u + tx \in \Omega$. By the continuity of the mapping $t \longmapsto u_t$, there exists $0 < s < 1$ such that $\|u_s\| < r$. By assumption, we have

$$s\langle y, x - x_0\rangle = \langle y, u_s - x_0\rangle \in (-\operatorname{int} C)^c \text{ and } \langle y,\ x - x_0\rangle \in (-\operatorname{int} C)^c.$$

The proof is complete. □

Theorem 2.2 *Let $\mathbb{K}$ be a closed convex cone in X, and let $x_0 \in \mathbb{K}_r$ with $r > 0$. If $\mathbb{K}_r^*(C, x_0) \setminus \mathbb{K}^*(C, x_0)$ contains some y_0, then $\|x_0\| = r$ and $\langle y_0, x_0\rangle \in \mathcal{Z} \setminus C$.*

Proof Since $\langle y_0, x_0 - x_0\rangle = 0$ and $y_0 \notin \mathbb{K}^*(C, x_0)$, by Theorem 2.1 we have $\|x_0\| = r$. Next, we prove that $\langle y_0, x_0\rangle \in \mathcal{Z} \setminus C$. Since $y_0 \in \mathbb{K}_r^*(C, x_0) \setminus \mathbb{K}^*(C, x_0)$, there exists $u_0 \in \mathbb{K} \setminus \mathbb{K}_r$ such that $\langle y_0, u_0 - x_0\rangle \in (-\operatorname{int} C)$. Choose a real number $t > 0$ such that $0 < t\|u_0\| < r$. Clearly, $0 < t < 1$ and $tu_0 \in \mathbb{K}_r$. If $\langle y_0, x_0\rangle \in C$, then

$$\langle y_0, tu_0 - x_0\rangle = t\langle y_0, u_0 - x_0\rangle - (1-t)\langle y_0, x_0\rangle \in (-\operatorname{int} C) - C \subset (-\operatorname{int} C).$$

This is a contradiction since $y_0 \in \mathbb{K}_r^*(C, x_0)$. Therefore, $\langle y_0, x_0\rangle \in \mathcal{Z} \setminus C$. □

3 Exceptional families of elements

In preparation of the definition of vectorial exceptional families of elements for multivalued mappings into $L(X, \mathcal{Z})$, we first recall the corresponding definition for multivalued mappings into X^*. Throughout this section, let $\mathbb{K}$ denote a closed convex cone in X, and let $J : X \longrightarrow 2^{X^*}$ denote the normalized duality mapping given by

$$J(x) = \{y \in X^* : \langle y, x\rangle = \|x\|^2 \text{ and } \|y\| = \|x\|\} \text{ for } x \in X.$$

For a given multivalued mapping T from $\mathbb{K}$ into X^* with nonempty values, according to [15], an exceptional family elements (in short, EFE) for $(T, \mathbb{K})$ is a subset of $\mathbb{K} \setminus \{0\}$ parameterized by positive numbers, say $\{x_r : r > 0\}$, satisfying the following conditions:

E1. $\lim_{r\to\infty} \|x_r\| = \infty$.
E2. For every $r > 0$, there exist a real number $\lambda_r > 0$, an $\widehat{y}_r \in J(x_r)$ and an $y_r \in T(x_r)$ such that $\lambda_r \widehat{y}_r + y_r \in \mathbb{K}^*(x_r)$.

As remarked in [4], given any $x \in X \setminus \{0\}$, by writing $\|x\| = \rho > 0$ the $\mathbb{R}_+$-dual cone of the closed ball $\mathbb{B}_\rho = \{u \in X : \|u\| \leq \rho\}$ at x is given by

$$\mathbb{B}^*_\rho(x) = \{ty : t \leq 0 \text{ and } y \in J(x)\}.$$

Thus, the condition **E2** given above can be restated as:

E2′. For every $r > 0$, there exist an $\widehat{y}_r \in -\mathbb{B}^*_{\rho(r)}(x_r) \setminus \{0\}$ and an $y_r \in T(x_r)$ such that $\widehat{y}_r + y_r \in \mathbb{K}^*(x_r)$, where $\rho(r) = \|x_r\|$.

This allows us to extend the definition of EFE to that for mappings into $L(X, \mathcal{Z})$.

Now, for a given multivalued mapping T from $\mathbb{K}$ into $L(X, \mathcal{Z})$ with nonempty values, a family $\{x_r : r > 0\}$ of elements of $\mathbb{K} \setminus \{0\}$ will be called a C-EFE for $(T, \mathbb{K})$ if the following conditions are satisfied:

VE1. $\lim_{r\to\infty} \|x_r\| = \infty$.
VE2. For every $r > 0$, there exist an $\widehat{y}_r \in -\mathbb{B}^*_{\rho(r)}(C, x_r) \setminus \{0\}$ and an $y_r \in T(x_r)$ such that $\langle \widehat{y}_r + y_r, x_r\rangle = 0$ and $\langle \widehat{y}_r + y_r, u\rangle \in C$ for all $u \in \mathbb{K}$, where $\rho(r) = \|x_r\|$.

EFE acceptable mappings Following [18], a multivalued mapping T as given above will be called C-EFE *acceptable* if either the problem MVCP$(T, \mathbb{K})$ has a solution, or there exists a C-EFE for $(T, \mathbb{K})$.

Now, we are ready to state and prove our main result which is a vectorial analogue of Theorem 3.1 in [4].

Theorem 3.1 *Assume that X is a Banach space. Given a multivalued mapping T from $\mathbb{K}$ into $L(X, \mathcal{Z})$ with nonempty values, if for every $r > 0$ the problem GVVI$(T, \mathbb{K}_r)$ has a solution, then T is C-EFE acceptable.*

Proof It suffices to show that there is a C-EFE for $(T, \mathbb{K})$ if the problem MVCP$(T, \mathbb{K})$ has no solutions. By assumption, for every $r > 0$ the problem GVVI$(T, \mathbb{K}_r)$ has a solution $(x_r, y_r) \in \mathcal{G}_T$ with $x_r \in \mathbb{K}_r$. Since the problem MVCP$(T, \mathbb{K})$ has no solutions, it follows that $y_r \in \mathbb{K}^*_r(C, x_r) \setminus \mathbb{K}^*(C, x_r)$. Now, from Theorem 3.2 given below we conclude that $\{x_r : r > 0\}$ is a C-EFE for $(T, \mathbb{K})$. □

Theorem 3.2 *Assume that X is a Banach space. Given a point $x_0 \in \mathbb{K}_r$ with $r > 0$, if $y_0 \in \mathbb{K}^*_r(C, x_0) \setminus \mathbb{K}^*(C, x_0)$, then $\|x_0\| = r$ and there exists $\widehat{y} \in -\mathbb{B}^*_r(C, x_0) \setminus \{0\}$ such that $\langle \widehat{y} + y_0, x_0\rangle = 0$ and $\langle \widehat{y} + y_0, u\rangle \in C$ for all $u \in \mathbb{K}$.*

Observe that for any $x \in \mathbb{K}$ and $y \in L(X, \mathcal{Z})$, if $\langle y, x\rangle = 0$ and $\langle y, u\rangle \in C$ for all $u \in \mathbb{K}$, then $y \in \mathbb{K}^*(C, x)$.

For the proof of Theorem 3.2, we need two lemmas. The first one is obtained by a slight modification of Theorem 3.7 in [8, Chapter IV, p. 113], and its proof is omitted.

Lemma 3.3 *Let A and B be disjoint nonempty convex subsets of $\mathcal{Z}$. If B is open, then there exist a non-zero linear functional $\varphi \in \mathcal{Z}^*$ and $\lambda \in \mathbb{R}$ such that $\langle \varphi, z\rangle < \lambda$ for all $z \in B$, and such that $\langle \varphi, z\rangle \geq \lambda$ for all $z \in A$.*

Lemma 3.4 *Given a point $x_0 \in \mathbb{K}_r$ with $r > 0$, if $y_0 \in \mathbb{K}_r^*(C, x_0) \setminus \mathbb{K}^*(C, x_0)$, then there is a non-zero functional $\varphi \in \mathcal{Z}^*$ such that*

(i) $\langle \varphi, z\rangle < 0$ *for all* $z \in (-\operatorname{int} C)$*;*
(ii) *the composition* $\varphi \circ y_0$ *lies in* $\mathbb{K}_r^*(x_0)$ *and* $\langle \varphi \circ y_0, x_0\rangle < 0$.

Proof Let $M_r = \{\langle y_0, u - x_0\rangle : u \in \mathbb{K}_r\}$. Note that M_r is a convex subset of $\mathcal{Z}$ with $M_r \cap (-\operatorname{int} C) = \emptyset$. By Lemma 3.3, there exist a non-zero linear functional $\varphi \in \mathcal{Z}^*$ and $\lambda \in \mathbb{R}$ such that $\langle \varphi, z\rangle < \lambda$ for all $z \in (-\operatorname{int} C)$, and such that $\langle \varphi, z\rangle \geq \lambda$ for all $z \in M_r$. We prove below that φ satisfies the conditions (i) and (ii).

By the definition, the linear functional $\varphi \circ y_0 \in X^*$ lies in $\mathbb{K}_r^*(x_0)$ if and only if $\langle \varphi \circ y_0, u - x_0\rangle \geq 0$ for all $u \in \mathbb{K}_r$, i.e., $\langle \varphi, z\rangle \geq 0$ for all $z \in M_r$. Thus, $\varphi \circ y_0 \in \mathbb{K}_r^*(x_0)$ whenever $\lambda = 0$. Choose an arbitrary point $z_0 \in (-\operatorname{int} C)$. For any real number $t > 0$, we have $tz_0 \in (-\operatorname{int} C)$ and $t\langle \varphi, z_0\rangle = \langle \varphi, tz_0\rangle < \lambda$. Letting $t \to 0^+$, we obtain $\lambda \geq 0$. On the other hand, since $0 = \langle y_0, x_0 - x_0\rangle \in M_r$, it follows that $\lambda \leq 0$ and $\lambda = 0$.

Finally, we prove that $\langle \varphi \circ y_0, x_0\rangle < 0$. Since $\varphi \circ y_0 \in \mathbb{K}_r^*(x_0)$, we have

$$0 \leq \langle \varphi \circ y_0, 0 - x_0\rangle = -\langle \varphi \circ y_0, x_0\rangle \text{ and } \langle \varphi \circ y_0, x_0\rangle \leq 0.$$

Suppose that $\langle \varphi \circ y_0, x_0\rangle = 0$. Then

$$\langle \varphi \circ y_0, u\rangle = \langle \varphi \circ y_0, u - x_0\rangle \geq 0 \text{ for all } u \in \mathbb{K}_r.$$

This implies that for every $u \in \mathbb{K}$,

$$\langle \varphi \circ y_0, u - x_0\rangle = \langle \varphi \circ y_0, u\rangle \geq 0 \text{ and } \langle y_0, u - x_0\rangle \in (-\operatorname{int} C)^c.$$

This is a contradiction to the assumption that $y_0 \notin \mathbb{K}^*(C, x_0)$. The proof is complete. □

Proof of Theorem 3.2 By Theorem 2.2, $\|x_0\| = r$. Let $\varphi \in \mathcal{Z}^*$ be given in Lemma 3.4, and choose an arbitrary point $e \in \operatorname{int} C$ with $\langle \varphi, e\rangle = 1$. Write $\langle \varphi \circ y_0, x_0\rangle = t_0 < 0$, and write $\mathbb{B}_r = \{x \in X : \|x\| \leq r\}$. Note that

$$\mathbb{K}_r^*(x_0) = \mathbb{K}^*(x_0) + \mathbb{B}_r^*(x_0);$$

see [2, Theorem 4.1.16, p. 174]. From Lemma 3.4 (ii), we conclude that there exist $\lambda \geq 0$ and $\sigma \in J(x_0)$ such that $\lambda\sigma + \varphi \circ y_0 \in \mathbb{K}^*(x_0)$, or equivalently,

$$\langle \lambda\sigma + \varphi \circ y_0, x_0\rangle = 0 \text{ and } \langle \lambda\sigma + \varphi \circ y_0, u\rangle \geq 0 \text{ for all } u \in \mathbb{K}.$$

Notice that $\lambda\sigma \in -\mathbb{B}_r^*(x_0) \setminus \{0\}$ since

$$0 = \langle \lambda\sigma + \varphi \circ y_0, x_0\rangle = \lambda r^2 + t_0 \text{ and } \lambda = -r^{-2} t_0 > 0.$$

Next, consider the continuous linear mapping $\ell : X \longrightarrow \mathcal{Z}$ defined by

$$\langle \ell, u\rangle = \frac{-1}{\lambda}(\langle y_0, u\rangle - \langle \varphi \circ y_0, u\rangle e) \text{ for } u \in X.$$

Observe that $\langle \ell, u\rangle \in \ker(\varphi)$ for all $u \in X$. Now, let $\widehat{y} = \lambda p$, where $p \in L(X, \mathcal{Z})$ is defined by

$$\langle p, u\rangle = \langle \ell, u\rangle + \langle \sigma, u\rangle e \text{ for } u \in X.$$

Then for every $u \in \mathbb{K}$,

$$\langle \widehat{y} + y_0, u\rangle = \langle \lambda\sigma + \varphi \circ y_0, u\rangle e \in C.$$

In particular,

$$\langle \widehat{y} + y_0, x_0\rangle = \langle \lambda\sigma + \varphi \circ y_0, x_0\rangle e = 0.$$

Finally, we prove that $\widehat{y} \in -\mathbb{B}^*_r(C, x_0) \setminus \{0\}$. Clearly, $\widehat{y} \neq 0$ since $\sigma \neq 0$. It remains to show that $\langle \widehat{y}, x_0 - u\rangle \in (-\operatorname{int} C)^c$ for all $u \in \mathbb{B}_r$. This will follow if

$$\lambda\langle \varphi \circ p, x_0 - u\rangle = \langle \varphi \circ \widehat{y}, x_0 - u\rangle \geq 0 \text{ for all } u \in \mathbb{B}_r;$$

see Lemma 3.4 (i). For $u \in \mathbb{B}_r$,

$$\langle \varphi \circ p, x_0 - u\rangle = \langle \sigma, x_0 - u\rangle = r^2 - \langle \sigma, u\rangle \geq r^2 - \|\sigma\|\|u\| = r(r - \|u\|) \geq 0$$

The proof is complete. □

Remark 3.5 Every single-valued mapping can be regarded as a multivalued mapping with every value a singleton. Therefore, previous results and definitions are valid naturally for single-valued mappings. More precisely, for a given mapping $T : \mathbb{K} \longrightarrow L(X, \mathcal{Z})$, let $\widehat{T}$ be the multivalued mapping given by $\widehat{T}(x) = \{T(x)\}$ for $x \in \mathbb{K}$.

(i) A family $\{x_r : r > 0\}$ of elements of $\mathbb{K} \setminus \{0\}$ is a C-EFE for $(T, \mathbb{K})$ if it is a C-EFE for $(\widehat{T}, \mathbb{K})$. Equivalently, $\lim_{r\to\infty} \|x_r\| = \infty$, and for every $r > 0$ there exists $\widehat{y}_r \in -\mathbb{B}^*_{\rho(r)}(C, x_r) \setminus \{0\}$ such that $\langle \widehat{y}_r + T(x_r), x_r\rangle = 0$ and $\langle \widehat{y}_r + T(x_r), u\rangle \in C$ for all $u \in \mathbb{K}$, where $\rho(r) = \|x_r\|$.
(ii) T is called C-EFE acceptable if $\widehat{T}$ is. Thus, when X is a Banach space, T is C-EFE acceptable if for every $r > 0$ the problem VVI$(T, \mathbb{K}_r)$ has a solution.

4 Pseudomonotone multivalued mappings

As an application of Theorem 3.1, in this section, we consider pseudomonotone multivalued mappings, and prove that, under some continuity assumptions, such a mapping from a closed convex cone in X into $L(X, \mathcal{Z})$ is C-EFE acceptable if X is a reflexive Banach space; see Theorem 4.5. The theorem will be proved by a similar argument as that given in the proof of Theorem 2.3.4 in [11].

To proceed, we recall some preliminary definitions. Throughout this section, all multivalued mappings are assumed to have nonempty values, and let Ω denote a nonempty *convex* subset of X. A multivalued mapping S from Ω into X is a KKM mapping if for every nonempty finite subset E of Ω, each element of co(E) lies in some $S(x)$ with $x \in E$, where co(E) denotes the convex hull of E.

A multivalued mapping T from Ω into $L(X, \mathcal{Z})$ is C-pseudomonotone if for any pair $(x, y) \in \mathcal{G}_T$, $\langle y, u - x\rangle \in (-\operatorname{int} C)^c$ implies $\langle w, u - x\rangle \in (-\operatorname{int} C)^c$ for all $(u, w) \in \mathcal{G}_T$.

A linear operator $\ell \in L(X, \mathcal{Z})$ is called completely continuous if for every weakly convergent sequence $\{x_n\}_{n=1}^{\infty}$ in X with the weak limit $x \in X$, the sequence $\{\langle \ell, x_n\rangle\}_{n=1}^{\infty}$ converges to $\langle \ell, x\rangle$ in the strong topology on $\mathcal{Z}$ [10]. When $\mathcal{Z}$ is finite dimensional, every element of $L(X, \mathcal{Z})$ is completely continuous.

To employ the arguments given in the proof of [11, Theorem 2.3.4], we first prove the following lemma.

Lemma 4.1 *Let Ω be a nonempty convex and weakly compact subset of X, let T be a multivalued mapping from Ω into $L(X, \mathcal{Z})$, and let Σ be the multivalued mapping from Ω into itself defined by*

$$\Sigma(u) = \{x \in \Omega : \langle w, u - x\rangle \in (-\operatorname{int} C)^c \text{ for all } w \in T(u)\}$$

for $u \in \Omega$. If T is C-pseudomonotone, and if for every $x \in \Omega$ all elements of $T(x)$ are completely continuous, then $\bigcap_{u\in\Omega} \Sigma(u) \neq \emptyset$.

Proof Note that Σ has nonempty values since $u \in \Sigma(u)$ for every $u \in \Omega$. By the weak compactness of Ω, it suffices to show that Σ has weakly closed values, and the family $\{\Sigma(u) : u \in \Omega\}$ has the finite intersection property.

To prove that $\Sigma(u)$ is weakly closed for every $u \in \Omega$, we consider an arbitrary $w \in T(u)$, and any sequence $\{x_n\}_{n=1}^{\infty}$ in $\Sigma(u)$ which converges weakly to some $x \in \Omega$. Since $(-\operatorname{int} C)^c$ is closed, and since w is completely continuous, we have

$$\langle w, u - x\rangle = \lim_{n\to\infty} \langle w, u - x_n\rangle \in (-\operatorname{int} C)^c,$$

i.e., $x \in \Sigma(u)$. Hence, $\Sigma(u)$ is weakly closed.

Next, we consider the multivalued mapping $S : \Omega \longrightarrow 2^{\Omega}$ defined by

$$S(u) = \{x \in \Omega : \langle y, u - x\rangle \in (-\operatorname{int} C)^c \text{ for some } y \in T(x)\}$$

for $u \in \Omega$. Clearly, $u \in S(u)$ for every $u \in \Omega$. Since T is C-pseudomonotone, it follows immediately from the definition that $S(u) \subset \Sigma(u)$ for every $u \in \Omega$, and that $\overline{S(u)} \subset \Sigma(u)$, where $\overline{S(u)}$ denotes the closure of $S(u)$. We shall complete the proof by showing that the family $\{\overline{S(u)} : u \in \Omega\}$ has the finite intersection property.

Observe that S is a KKM mapping; see e.g., [9] for a proof. For any nonempty finite subset E of Ω, we write $A = \operatorname{co}(E)$ and $S_A(u) = A \cap S(u)$ for $u \in A$. Since the mapping $u \longmapsto \overline{S_A(u)}$ is also a KKM mapping, from Fan-KKM Theorem [12] we conclude that

$$\bigcap_{u\in E} \overline{S(u)} \supset \bigcap_{u\in A} \overline{S_A(u)} \neq \emptyset.$$

This proves that the family $\{\overline{S(u)} : u \in \Omega\}$ has the finite intersection property. □

To establish our results, we have to impose some continuity conditions on multivalued mappings into $L(X, \mathcal{Z})$. First, we need the simple convergence topology on $L(X, \mathcal{Z})$ which is the unique translation-invariant topology having the family

$$\{[x, V] : x \in X \text{ and } V \text{ is 0-neighborhood in } \mathcal{Z}\}$$

as its 0-neighborhood base, where

$$[x, V] = \{y \in L(X, \mathcal{Z}) : \langle y, x\rangle \in V\}.$$

When $\mathcal{Z} = \mathbb{R}$, this topology coincides with the weak-star topology on X^*. See [23] for a full discussion on the simple convergence topology. We shall use $L_s(X, \mathcal{Z})$ for $L(X, \mathcal{Z})$ endowed with the simple convergence topology. A net $\{y_\alpha\}$ in $L(X, \mathcal{Z})$ is called convergent in $L_s(X, \mathcal{Z})$ to some $y \in L(X, \mathcal{Z})$, written by $y_\alpha \xrightarrow{s} y$, if for every point $x \in X$ the net $\{\langle y_\alpha, x\rangle\}$ converges to $\langle y, x\rangle$ in the strong topology on $\mathcal{Z}$.

Recall that a multivalued mapping T from a topological space W into a topological space Y is upper semicontinuous at $w \in W$ if for every open set $G \subset Y$ containing $T(w)$ there is an open set $N \subset W$ containing w such that $T(w') \subset G$ whenever $w' \in N$. If T is upper semicontinuous at every point of W, then T is simply called upper semicontinuous.

Hemicontinuity. A multivalued mapping T from Ω into $L_s(X, \mathcal{Z})$ is hemicontinuous if for any points $x_0, x_1 \in \Omega$ the multivalued mapping $t \longmapsto T(x_t)$ is upper semicontinuous from the interval [0, 1] into $L_s(X, \mathcal{Z})$, where $x_t = (1-t)x_0 + tx_1$ for $0 \le t \le 1$.

By a standard argument, one proves easily that an upper semicontinuous multivalued mapping into $L_s(X, \mathcal{Z})$ is hemicontinuous.

Regular hemicontinuity. A multivalued mapping T from Ω into $L_s(X, \mathcal{Z})$ will be called regularly hemicontinuous if for any points $u, x \in \Omega$ the following condition is satisfied:

(*) If $u_t = (1-t)x + tu$ and $w(t) \in T(u_t)$ for $0 \le t \le 1$, then there is a net $\{t_\alpha\}$ with $0 < t_\alpha < 1$ and $t_\alpha \longrightarrow 0$ such that $w(t_\alpha) \xrightarrow{s} y$ for some $y \in T(x)$.

When T is single-valued, the above condition $(*)$ becomes:

$(**)$ If $u_t = (1-t)x + tu$ for $0 \le t \le 1$, then there is a net $\{t_\alpha\}$ with $0 < t_\alpha < 1$ and $t_\alpha \longrightarrow 0$ such that $T(u_{t_\alpha}) \xrightarrow{s} T(x)$.

Theorem 4.2 *Let T be a multivalued mapping from Ω into $L_s(X, \mathcal{Z})$. If T is hemicontinuous and has compact values, then T is regularly hemicontinuous.*

Proof For any points $u, x \in \Omega$ and real numbers $0 \le t \le 1$, let u_t and $w(t)$ be given above. Choose any sequence $\{t_n\}_{n=1}^{\infty}$ of numbers with $0 < t_n < 1$ for every n and $\lim_{n\to\infty} t_n = 0$. For every integer $n > 0$, we write $w(t_n) = w_n$.

Let Φ be the multivalued mapping from [0, 1] into $L_s(X, \mathcal{Z})$ defined by $\Phi(t) = T(u_t)$ for $0 \le t \le 1$. Since Φ is upper semicontinuous and has compact values, it follows from [3, Theorem VI.3, p. 110] that the image

$$\bigcup_{0 \le t \le 1} \Phi(t)$$

of Φ is a compact subset of $L_s(X, \mathcal{Z})$. It now follows from [1, Theorem 16.17, p. 532] that $\{w_n\}_{n=1}^{\infty}$ has a subnet which converges in $L_s(X, \mathcal{Z})$ to some $y \in \Phi(0) = T(x)$. □

Theorem 4.3 *Let Ω be a nonempty convex and weakly compact subset of X, and let T be a multivalued mapping from Ω into $L_s(X, \mathcal{Z})$ with compact values. Assume that for every $x \in \Omega$ all elements of $T(x)$ are completely continuous. If T is C-pseudomonotone and regularly hemicontinuous, then the problem GVVI(T, Ω) has a solution.*

Theorem 4.3 follows immediately from Lemma 4.1 and the next theorem.

Theorem 4.4 *Let Ω be a nonempty closed and convex subset of X, and let T be a regularly hemicontinuous multivalued mapping from Ω into $L_s(X, \mathcal{Z})$ with compact values. If there exists $\widehat{x} \in \Omega$ such that $\langle w, u - \widehat{x}\rangle \in (-\operatorname{int} C)^c$ for all $(u, w) \in \mathcal{G}_T$, then the problem GVVI(T, Ω) has a solution.*

Proof For any given $u \in \Omega$ and $0 \le t \le 1$, write $u_t = (1-t)\widehat{x} + tu$, and choose an arbitrary $w(t) \in T(u_t)$. By assumption, there is a net $\{t_\alpha\}$ with $0 < t_\alpha < 1$ for every α and $t_\alpha \longrightarrow 0$ such that $w(t_\alpha) \xrightarrow{s} \widehat{y}$ for some $\widehat{y} \in T(\widehat{x})$. Since for every α,

$$t_\alpha \langle w(t_\alpha), u - \widehat{x}\rangle = \langle w(t_\alpha),\ u_{t_\alpha} - \widehat{x}\rangle \in (-\operatorname{int} C)^c,$$

and since $(-\operatorname{int} C)^c$ is closed, we have $\langle \widehat{y}, u - \widehat{x}\rangle = \lim_\alpha \langle w(t_\alpha), u - \widehat{x}\rangle \in (-\operatorname{int} C)^c$. □

Theorem 4.5 *Let $\mathbb{K}$ be a closed convex cone in a reflexive Banach space X, and let T be a multivalued mapping from $\mathbb{K}$ into $L_s(X, \mathcal{Z})$ with compact values. Assume that for every $x \in \mathbb{K}$ all elements of $T(x)$ are completely continuous. If T is C-pseudomonotone and regularly hemicontinuous, then T is C-EFE acceptable.*

Proof For every $r > 0$, the set $\mathbb{K}_r$ is weakly compact in X. By Theorem 4.3, the problem GVVI$(T, \mathbb{K}_r)$ has a solution. Now, the proof is completed by Theorem 3.1. □

We end this paper by making a quick comparison between Theorem 4.3 given above and Theorem 1 of [9]. First, in Theorem 4.3, the continuity assumption on the mapping T is slightly weaker than the hemicontinuity assumed in [9]. Secondly, in [9, Theorem 1], each value of T is required to be compact in the norm topology on $L(X, \mathcal{Z})$ which is much stronger than weak* compactness as imposed in Theorem 4.3. However, Daniilidis and Hadjisavvas' result holds for mappings with a weaker pseudomonotonicity.

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